Available online at www.sciencedirect.com

ScienceDirect

Materials Today: Proceedings XX (2016) XXX–XXX

materialstoday: PROCEEDINGS

www.materialstoday.com/proceedings

PMME 2016

Bayesian Cramer-Rao Lower Bound of Variances Under Ranked Set Sampling

Vivek Verma[a] * , Dilip C. Nath[a], Radhakanta Das[b]

[a]*Department of Statistics,Gauhati University, Guwahati 781014, Assam, India*
[b]*Department of Statistics, Presidency University,86/1 College Street Kolkata, 700073, West Bengal, India*

Abstract

Ranked set sampling (RSS) is regarded as a substitute of simple random sampling (SRS). Under SRS, obtained samples are only independent, whereas in cases of balanced RSS, observations are independent order statistics. This paper presents a method to calculate the Bayesian Cramer Rao (BCR) bound when the statistical model is based on independent order statistics. This bound is also compared with independent simple random samples. In context of order samples where this classical Cramer Rao (CR) bound is hard to evaluate, this study presents a closed-form expression of a BCR bound. The efficiency in BCR bound due to independent order observations as compared to independent samples, is procure on the basis of theoretical and simulation results. Through this one can evaluate the significance of using independent order statistics as compare to independent samples for the gains in accuracy to estimate the parameters and reduction in required sample size. The procured results demonstrate that BCR bound based on independent order statistics is more compact, efficient observational economic than SRS based bound of estimator.

Selection and Peer-review under responsibility of International Conference on Processing of Materials, Minerals and Energy (July 29th – 30th) 2016, Ongole, Andhra Pradesh, India.

Keywords: Order Statistics; Fisher's Information, Relative efficiency, Cramer-Rao Bound

* Corresponding author. Tel.: +91-8486650289 fax: +0-000-000-0000 .
E-mail address: viv_verma456@yahoo.com

Selection and Peer-review under responsibility of International Conference on Processing of Materials, Minerals and Energy (July 29th – 30th) 2016, Ongole, Andhra Pradesh, India.

1. Introduction

McIntyre's (1952) proposed method of Ranked Set Sampling (RSS), is found to more efficient than Simple Random Sampling (SRS) for estimation of unknown parameters of continuous distributions. The applicability of RSS is more, especially in those studies where auxiliary information on a sample unit is more economical and can be ordered quantified as compared to original variable. The procedure of RSS can be illustrated as follows: firstly, a random sample of size s^2 is drawn from the target population and are segregated in s sets each of consists of s simple random samples. Secondly, each set of units is ranked in ascending order by using the variable which is economical and easily available. And finally, the element with lowest rank of the first set is selected for measuring the variable of interest, as is the second lowest element of from the second set are selected. This process is continued until the s^{th} highest ranked element from the s^{th} set is measured. If this entire process is repeated m times, then we have a ranked set sample of size $n = ms$.

Since, SRS yields the samples which are independent and identically distributed, whereas observations of RSS are independent non identical order statistics. To measure the amount of information of the samples obtained through RSS as compare to SRS the Fisher information matrix have already been taken into consideration by Stokes(1995), Barabesi and El-Sharaawi, (2001) and Chen(2000). In these works, the aim was to compare the variability level in the estimates of the unknown parameters of interest under both of the sampling. In recent work of Hatefi and Jozani(2013), Frey(2014) and Biradar and Santosha(2014), Fisher information has derived for different situations and multi-parameter distributional set-up under RSS.

The classical Cramer-Rao(CR) bound which had been derived by using the Fisher's information, provides an inequality for accuracy of parameter estimators under certain regularity conditions. The derivation of the CR bound is possible if the estimator is unbiased and the parameter of interest is non-random. In practice, it is not always possible to establish such an inequality where both the conditions hold. In case of RSS, observations are order dependent (Dell and Clutter, 1972), and estimators are mostly biased (Chen et.al.,(2004), Muttlak and Abu-Dayyeh (2004), Bouza (2005), Bhoj and Kushary (2014)). In addition of that if the parameters are random variable under RSS based models, then assessment of the estimator becomes a big question and is not considered in the previous literatures.

Derivation of the CR bound by using independent ordered samples of RSS, where the parameter is considered as a random variable, is indispensable. Therefore, present work discusses a promising framework to address this situation by deriving the Bayesian Cramer-Rao (BCR) bound, where the parameter is also a random variable. This is quite similar to the classical CR bound, but does not depend on their assumptions.

This paper presents a method to derive and calculate the BCR bound for the parameter of a continuous family of distribution under RSS, where ranked set samples are independent order statistics. In addition to that the expression for the interrelationship among BCR bounds under both SRS and RSS, is also derived. Through this relationship an attempt is being made to discuss the effectiveness of using independent ordered samples as compare to independent samples, for obtaining information about the parameter. It is found in both of the theoretical and simulation results that BCR bound to a parameter under RSS is more compact and efficient than that of SRS based bounds. In any sampling procedure sample size is always be a major concern. Since, RSS is regraded as a observation economical and superior to ordinary sampling, therefore, illustration of the performance is also discussed in terms of sample size reduction.

2. Bounds for parameter

2.1. Cramer-Rao Bound for a non-random parameter

Let $X_1, X_2, \cdots, X_n$ are n independent and identically distributed *(iid)* random variable from an absolutely continuous probability density function(*pdf*) $f(x|\theta)$ and distribution function $F(x|\theta)$, where $\theta \in (a, b) \subset R$, denotes the parameter of interest. Let s simple random samples $\mathbf{x_{SRS}}=(x_1, x_2,...,x_n)$of size n drawn from the population $f(x|\theta)$, and the likelihood function of the $\mathbf{x_{SRS}}$, which is the probability of the observed data for the given parameter θ, is given by

$$L(\boldsymbol{x_{SRS}}|\theta) = \prod_{i=1}^{n} f(x_i|\theta)\ ;\ \theta \in (a,b) \tag{1}$$

Under balanced RSS, n ranked set samples are obtained by quantifying the samples through judgement into s independent, non identical ordered sets, where each i^{th} judgment order statistic consists of m *iid* samples. Let $X_{[i]j}$'s ; $i = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, m$, are n (= ms) balanced ranked set samples for fixed set size s, and replications m, from the same family of distribution. The *pdf* of $X_{[i]j}$ which depends on the i^{th} order statistics is given by

$$f_{(i)}(x|\theta) = s\binom{s-1}{i-1}\left(F(x|\theta)\right)^{i-1}\left(1-F(x|\theta)\right)^{s-i} f(x|\theta)\ ;\ i = 1,2,\dots,s \tag{2}$$

Let $\mathbf{x_{RSS}} = (\mathbf{x}_{[1]}, \cdots, \mathbf{x}_{[i]}, \cdots, \mathbf{x}_{[s]})$ where $\mathbf{x}_{[i]} = (x_{[i]1}, \cdots, x_{[i]m})^T$, be (m, s) dimensional matrix of independent ranked set samples. It is to be emphasized under RSS that each $\mathbf{X}_{[i]}$ are independent and non-identically distributed, although for fixed i^{th} set, $X_{[i]1}, \cdots, X_{[i]m}$ are independent and identically distributed. The joint density function of $\mathbf{X}_{[i]}$ for given the parameter θ will be obtained as

$$f\left(\boldsymbol{x}_{[i]}|\theta\right) = \prod_{j=1}^{m} f_{(i)}\left(x_{[i]j}|\theta\right) \tag{3}$$

and the likelihood function of $\boldsymbol{x_{RSS}}$ given the same parameter θ, is of the form:

$$L(\boldsymbol{x_{RSS}}|\theta) = \prod_{i=1}^{s} f\left(\boldsymbol{x}_{[i]}|\theta\right) = \prod_{i=1}^{s}\prod_{j=1}^{m} f_{(i)}\left(x_{[i]j}|\theta\right) \tag{4}$$

In the first part of this section, interest lies in the information contained in samples of SRS and RSS, $\boldsymbol{x_{SRS}}$ and $\boldsymbol{x_{RSS}}$, respectively, about the non-random parameter θ. Let $\hat{\theta}(\boldsymbol{x_{SRS}})$ and $\hat{\theta}(\boldsymbol{x_{RSS}})$, are maximum likelihood estimates, and are considered for providing information to the highest accuracy of an unknown parameter, based on $\boldsymbol{x_{SRS}}$ and $\boldsymbol{x_{RSS}}$ respectively.

Let $I_{SRS}(\theta)$and $I_{RSS}(\theta)$, denotes the amount of information carried by samples of SRS and RSS, respectively, about an unknown parameter $\theta \in (a, b)$, and are obtained using the relations

$$I_{SRS}(\theta) = \int_{\mathcal{R}^n}\left(\frac{\delta log\, L(\boldsymbol{x_{SRS}}|\theta)}{\delta\theta}\right)^2 L(\boldsymbol{x_{SRS}}|\theta)\, d\,\boldsymbol{x_{SRS}} \tag{5}$$

$$I_{RSS}(\theta) = \int_{\mathcal{R}^n}\left(\frac{\delta log\, L(\boldsymbol{x_{RSS}}|\theta)}{\delta\theta}\right)^2 L(\boldsymbol{x_{RSS}}|\theta)\, d\,\boldsymbol{x_{RSS}} \tag{6}$$

Theorem-1: Let $\hat{\theta}(\boldsymbol{x_{RSS}})$ be an estimator of θ based on ranked set samples. Under the assumption that θ is non-random parameter the Fisher information derived by Barabesi and El-Sharaawi (2001) is of the form

$$I_{RSS}(\theta) = I_{SRS}(\theta) + I_R(\theta) \tag{7}$$

$$I_{RSS}(\theta) \geq I_{SRS}(\theta) \tag{8}$$

where,

$$I_R(\theta) = s(s-1)E_{xRSS\,|\theta}\left(\left(\frac{\delta F(x|\theta)}{\delta\theta}\right)^2\left[F(x|\theta)\left(1-F(x|\theta)\right)\right]^{-1}\right)$$

$$= -\,s(s-1)E_{xRSS\,|\theta}\left(\frac{\delta F(x|\theta)}{\delta\theta}\,\frac{\delta\left(1-F(x|\theta)\right)}{\delta\theta}\right) \tag{9}$$

where $I_R(\theta)$, symbolize the additional information stored in ranked set samples due to the ranks regarding the parameter.

The classical CR bound, which is a generally considered as a measure of accuracy, provides a lower bound to the variance of estimators of a parameter. By using the equations (7) and inequality (8) of **Theorem-1** , the CR bound of $\hat{\theta}(\boldsymbol{x_{SRS}})$ and $\hat{\theta}(\boldsymbol{x_{RSS}})$, will be of the forms,

$$V\left(\hat{\theta}(\boldsymbol{x_{SRS}})\right) \geq {I_{SRS}}^{-1}(\theta) \geq 0 \tag{10}$$

$$V\left(\hat{\theta}(\boldsymbol{x_{RSS}})\right) \geq {I_{RSS}}^{-1}(\theta) \geq 0 \tag{11}$$

$${I_{SRS}}^{-1}(\theta) \geq {I_{RSS}}^{-1}(\theta) \tag{12}$$

$$\Rightarrow V\left(\hat{\theta}(\boldsymbol{x_{SRS}})\right) - V\left(\hat{\theta}(\boldsymbol{x_{RSS}})\right) \geq 0 \tag{13}$$

Equation (13) implies that the classical CR bound for the variance of estimator of the parameter θ based on ranked set samples, $\hat{\theta}(\boldsymbol{x_{RSS}})$, will be more compact than SRS based estimator.

2.2 Bayesian Cramer-Rao Bound for a random parameter

In the previous part, the Fisher information measure and CR bound of the parameter estimators are considered. To this end it is assumed that the parameter is a non-random variable. The aim of this section is to consider the most practical situations where θ is a random variable. To Bayesianise this set-up of θ, a prior distribution on the parameter space Θ have assumed, and denoted as $g(\theta)$. In order to obtain the information when the parameter is assumed as a random variable, by using samples $\boldsymbol{x_{SRS}}$ and $\boldsymbol{x_{RSS}}$, the BCR bounds of the parameter estimates are derived.

Let $E_{xRSS\,|\theta}$ denote the expectation with respect to conditional distribution of $(\boldsymbol{x_{RSS}}|\theta)$ and E_θ denotes the expectations over the density of the θ, g.

Assumption-1:

$$\int_a^b \frac{\delta log\, L(\boldsymbol{x_{RSS}}|\theta)}{\delta\theta} d\theta = 0 \tag{14}$$

Assumption-2: θ is defined over the compact interval (a,b) and $g(y) \to 0$ as $y \to a$ and $y \to b$, so that $g(a) = g(b) = 0$.

Theorem-2: Suppose $\hat{\theta}(\boldsymbol{x_{RSS}})$ as an estimate of θ based on ranked set samples and assumptions 1 and 2 are satisfied then

$$E_\theta\left[E_{xRSS\,|\theta}\left(\hat{\theta}(\boldsymbol{x_{RSS}}) - \theta\right)^2\right] \geq \left[E_\theta\left(I_{RSS}(\theta)\right) + I_g(\theta)\right]^{-1} \tag{15}$$

$$I_g(\theta) = \int \left(\frac{\delta \log\, g(\theta)}{\delta\theta}\right)^2 g(\theta) d\theta$$

Proof: The proof of Theorem-2 is given in Appendix.

For same choice of prior $g(\theta)$ for parameter if the samples, $\mathbf{x}_{SRS}$, are obtained through the procedure of SRS then Gill and Levit (1995) and Stoica and Ng (1998) derived the BCR bound which is given by

$$E_\theta\left[E_{xSRS\,|\theta}\left(\hat{\theta}(\boldsymbol{x_{SRS}}) - \theta\right)^2\right] \geq \left[E_\theta\left(I_{SRS}(\theta)\right) + I_g(\theta)\right]^{-1} \tag{16}$$

From equation-(7) and (A-9) of Appendix we have

$$E_\theta\left(I_{RSS}(\theta)\right) = E_\theta\left(I_{SRS}(\theta) + I_R(\theta)\right) = E_\theta\left(I_{SRS}(\theta)\right) + E_\theta\left(I_R(\theta)\right)$$

$$E_\theta\left(I_{RSS}(\theta)\right) + I_g(\theta) = E_\theta\left(I_{SRS}(\theta)\right) + E_\theta\left(I_R(\theta)\right) + I_g(\theta)$$

$$\geq E_\theta\left(I_{SRS}(\theta)\right) + I_g(\theta)$$

$$\left[E_\theta\left(I_{RSS}(\theta)\right) + I_g(\theta)\right]^{-1} \leq \left[E_\theta\left(I_{SRS}(\theta)\right) + I_g(\theta)\right]^{-1} \tag{17}$$

By using the BCR bounds of equations-(15-16) and the relation of equation-(17) it is found that

$$E_\theta\left[E_{xRSS\,|\theta}\left(\hat{\theta}(\boldsymbol{x_{RSS}}) - \theta\right)^2\right] \leq E_\theta\left[E_{xSRS\,|\theta}\left(\hat{\theta}(\boldsymbol{x_{SRS}}) - \theta\right)^2\right] \tag{18}$$

3. Evolution of Performance of Bounds

In this section, performance of both classical CR and Bayesian CR bounds are evaluated in terms of their relative efficiency (RE), reduction in sample size.

3.1 Relative Efficiency

The relative efficiency of θ based on above equation may be defined as

$$RE(\theta) = \frac{E_\theta\left[E_{xSRS\,|\theta}\left(\hat{\theta}(\boldsymbol{x_{SRS}})-\theta\right)^2\right]}{E_\theta\left[E_{xRSS\,|\theta}\left(\hat{\theta}(\boldsymbol{x_{RSS}})-\theta\right)^2\right]} \geq 1$$

$$= \frac{E_\theta\left(I_{RSS}(\theta)\right)+I_g(\theta)}{E_\theta\left(I_{SRS}(\theta)\right)+I_g(\theta)} \geq \frac{E_\theta\left(I_{RSS}(\theta)\right)}{E_\theta\left(I_{SRS}(\theta)\right)} \geq 1 \tag{19}$$

since, $I_g(\theta) > 0$. From equation (17)

$$\frac{E_\theta\left(I_{RSS}(\theta)\right)}{E_\theta\left(I_{SRS}(\theta)\right)} = \frac{E_\theta\left(I_{SRS}(\theta)+I_R(\theta)\right)}{E_\theta\left(I_{SRS}(\theta)\right)} = 1 + \frac{E_\theta\left(I_R(\theta)\right)}{E_\theta\left(I_{SRS}(\theta)\right)} \tag{20}$$

From the above equation (20) the following points are justified:

i) if ranking is perfect and obtained sample information $I_R(\theta)$, regarding parameter $\varepsilon > 1$, times more as compare to $E_\theta\left(I_{SRS}(\theta)\right)$ (independent of θ) then $RE(\theta) > 1 + \varepsilon$

ii) if ranking is perfect or imperfect and obtained sample information $I_R(\theta)$, regarding parameter is $0 < \gamma < 1$, times less as compare to $E_\theta\left(I_{SRS}(\theta)\right)$ then, $RE(\theta) > 1 + \gamma$

iii) if ranking is perfect and obtained sample information $I_R(\theta)$, regarding parameter is equal to $E_\theta\left(I_{SRS}(\theta)\right)$ then $RE(\theta) = 2$.

In all the above discussed conditions and on the basis of equations-(8),(13), (15) and (18) we find that the BCR bound of the parameter based on ranked set sample will always be more compact as compared to the other bounds.

3.2 Reduction of Sample Size

To determine the reduction in sample size due to RSS as of SRS in estimation of the parameter and minimum sample size required in RSS to achieve the same precision as of SRS, for the same prior choice of the parameter respectively, are calculated as

$$\%\, Reduction\ in\ sample\ size = \left(1 - \frac{E_\theta\left[E_{xRS\ S|\theta}\left(\hat{\theta}(\boldsymbol{x_{RSS}}) - \theta\right)^2\right]}{E_\theta\left[E_{xSRS\ |\theta}\left(\hat{\theta}(\boldsymbol{x_{SRS}}) - \theta\right)^2\right]}\right) X\ 100 \quad (21)$$

$$Required\ Sample\ Size\ = n\ \frac{E_\theta\left[E_{xRSS\ |\theta}\left(\hat{\theta}(\boldsymbol{x_{RSS}}) - \theta\right)^2\right]}{E_\theta\left[E_{xSRS\ |\theta}\left(\hat{\theta}(\boldsymbol{x_{SRS}}) - \theta\right)^2\right]} \quad (22)$$

4. Example

Let X be an absolute continuous Exponential random variable with $pdf\ f(x|\lambda) = \lambda e^{-\lambda x}$ and corresponding distribution function $F(x|\lambda) = \left(1 - e^{-\lambda x}\right)$ with parameter $\lambda > 0$. Let $f_{(r)}(x|\lambda)$ be the distribution function of r^{th} order statistics of exponential sample having size s then equation-(3) implies

$$f_{(r)}(x|\lambda) = s\binom{s-1}{r-1}\left(1 - e^{-\lambda x}\right)^{r-1}\lambda e^{-\lambda x}\ (e^{-\lambda x})^{s-r}$$

Suppose a ranked set samples having for fixed set size s and one cycle *i.e.*, $m = 1$ is considered, and are denotes as $X_{[1]}, X_{[2]}, \cdots, X_{[s]}$. Here, $X_{[r]}$'s for $r = 1, 2, \cdots, s$ are independent random variables with $pdf\ f_{(r)}(x|\lambda)$ and also let λ be a random variable having $Gamma(a,b)$ prior probability density functions, then

$$L(\boldsymbol{X_{RSS}}|\lambda) = \prod_{r=1}^{s} s\binom{s-1}{r-1}\left(1 - e^{-\lambda x}\right)^{r-1}\lambda e^{-\lambda x}\ (e^{-\lambda x})^{s-r}$$

$$g(\lambda|a,b) = \frac{b^a\lambda^{b-1}e^{-\lambda b}}{\Gamma(a)}; a, b > 0 \quad (23)$$

The classical Fisher information of parameter λ, $I_{SRS}(\lambda)$ and $I_{RSS}(\lambda)$ under SRS and RSS respectively are obtained using equations-(7-9)

$$I_{SRS}(\lambda) = \frac{s}{\lambda^{\mathbf{2}}}$$

$$I_{RSS}(\lambda) = I_{SRS}(\lambda) + I_R(\lambda)$$

$$I_R(\lambda) = s(s-1)E_{x|\lambda}\left[\frac{x^2 e^{-\lambda x}}{1 - e^{-\lambda x}}\right]$$

The Bayesian Fisher information of parameter λ, under SRS and RSS and their relative efficiency are given by

$$RE(\lambda) = \frac{E_\lambda(I_{RSS}(\lambda))}{E_\lambda(I_{SRS}(\lambda))} = 1 + \frac{E_\lambda(I_R(\lambda))}{E_\lambda(I_{SRS}(\lambda))} = 1 + (s-1)\frac{E_\lambda(\lambda^{-2})}{E_\lambda E_{x|\lambda}\left[\frac{x^2 e^{-\lambda x}}{1 - e^{-\lambda x}}\right]} \quad (24)$$

Let

$$\frac{E_\lambda\left(\frac{1}{\lambda^2}\right)}{E_\lambda E_{x|\lambda}\left[\frac{\lambda x^2 e^{-\lambda x}}{1 - e^{-\lambda x}}\right]} = \left(\int_0^\infty \lambda^{-2}\ g(\lambda|a,b)\ d\lambda\right)\left(\int_0^\infty\int_0^\infty \frac{x^2 e^{-\lambda x}}{1 - e^{-\lambda x}}\ f(x|\lambda)g(\lambda|a,b)dxd\lambda\right)^{-1} = \delta(a,b)$$

(25)

where $\delta(a,b)$ denotes the function which depends on the values of hyper-parameters $a, b > 0$ only. The relative efficiency of λ using equation-(25) is of the form

$$RE(\lambda) = 1 + (s - 1)\delta(a, b) \quad (27)$$

To illustrate the usefulness of the BCR bound of variance, based on both SRS and RSS derived in the preceding sections, simulation is done for the exponential distribution having parameter λ. Simulations are done for the cases where $a, b = \{1, 3, 5, 7, 9\}$ and $s = \{3, 5, 10\}$. The relative efficiency, reduction in sample size (in %) and required sample size for parameter λ, under RSS as compare to SRS, based on simulation results, for all cases are presented in Tables 1, 2 and 3 of Appendix section. From Table 1-3, it is observed that the BCR bound under RSS incorporates the effect of the prior and is a valid lower bound as compare to SRS based bounds. It is also observed that with increase in set size, s, the relative efficiencies are increasing and results to reduction in required sample size.

The pattern of change in the relative efficiency, reduction in sample size (in %) and required sample size, of the parameter estimator with change in values of a and b, are also depicted in Figure 1. This shows the pattern of change in relative efficiency for different combinations of $a \in (0, 10)$ and $b \in (0, 10)$ for set sizes $s = \{3, 5, 10\}$. It also shows that the sample of RSS, which are sets of independent ordered observations, provides more accurate and effective parameter estimates even in cases where the parameter is a random quantity.

5. Conclusion

In this study, a closed form expression of the BCR bound of the variance is derived for the design of RSS and compared with the classical SRS based bound. When the unknown parameter is assumed as a non-random quantity, one can derive that the estimate obtained under RSS are more efficient than SRS, by using the earlier work of Barabesi and El-Sharaawi (2001). For situations where the parameter is considered as a random variable, an attempt was made in this study to derive the BCR bounds using the ranked set samples. The relationship between the BCR bounds obtained by using the samples of SRS and RSS, and for fixed prior distribution of the parameter, also derived. It is found in both theory and simulated results that BCR bound of a parameter estimate under RSS provides not only a very compact bound with higher efficiency than that SRS based estimates but also very observation economic.

Acknowledgements

The first author like express his deepest thank to Department of Science & Technology, India respectively, for providing financial assistance. The second author like express his deepest thank to UGC-BSR for the grant support.

References

[1] Barabesi, L., and El-Sharaawi, A. (2001). The efficiency of ranked set sampling for parameter estimation, Statistics & Probability Letters. **53** 189–199.

[2] Bhoj, S. D. and Kushary, D. (2014). Ranked Set Sampling with Unequal Samples and Unequal Replications, Pak. J. Statist., **30(3)** 361–372.

[3] Biradar, B.S., and Santosha, C.D. (2014). Estimation of the Mean of the Exponential Distribution Using Maximum Ranked Set Sampling with Unequal Samples. Open Journal of Statistics, **4** 641-649.

[4] Bouza, N. C. (2005). Sampling using ranked sets: Concepts, results and perspectives. Revista Investigacion Operacional, **26(3)** 275–293.

[5] Chen, Z. (2000). The efficiency of ranked-set sampling relative to simple random sampling under multi-parameter families. Statistica Sinica **10** 247–263.

[6] Chen, Z., Bai, Z.D. and Sinha, B.K. (2004). Ranked Set Sampling: Theory and Applications, Lecture Notes in Statistics, 176,Springer-Verlag, New York.

[7] Dell, T . R. and Clutter, J. L. (1972). Ranked set sampling theory with order statistics background. Biometrics, **28** 545–555.

[8] Frey, J. (2014), A Note on Fisher Information and Imperfect Ranked-Set Sampling. Communications in Statistics: Theory and Methods, **43(14)** 2726–2733.

[9] Gill, D. R. and Levit Y. B. (1995). Applications of the van Trees Inequality: A Bayesian Cramer-Rao Bound. Bernoulli, **1(1/2)** 059–079.

[10] Hatefi, A., and Jozani, J. M.(2013). Fisher information in different types of perfect and imperfect ranked set samples from finite mixture models. Journal of Multivariate Analysis, **119** 16–31.

[11] McIntyre, G. A. (1952). A method for unbiased selective sampling, using ranked sets. Australian Journal of Agricultural Research, **3** 385–390.
[12] Muttlak, H.A. and Abu-Dayyeh, W.A. (2004). Weighted modified ranked set sampling. Applied Mathematics and Computation, 151 645–657.
[13] Stoica, P., and Ng C. B. (1998). On the Cramer-Rao Bound Under Parametric Constraints. IEEE SIGNAL PROCESSING LETTERS, **5(7)** 177–179.
[14] Stokes, L. (1995). Parametric ranked set sampling. Annals of the Institute of Statistical Mathematics, **47** 465–482.

Appendix

Proof of Theorem-2: Under the assumption-2,

$$\int_a^b \left(\frac{\delta}{\delta\theta} L(X_{RSS} \mid \theta) g(\theta) \right) d\theta = L(X_{RSS} \mid \theta) g(\theta) \mid_a^b = 0,$$

that implies

$$\int_{R^n} \hat{\theta}(X_{Rss}) \int_a^b \left(\frac{\delta}{\delta\theta} L(X_{RSS} \mid \theta) g(\theta) \right) d\theta dX_{RSS} = 0 \qquad \text{(A-1)}$$

Now let us consider the equation

$$\int_{R^n} \int_a^b \theta \left(\frac{\delta}{\delta\theta} L(X_{RSS} \mid \theta) g(\theta) \right) d\theta dX_{RSS} \qquad \text{(A-2)}$$

and by using the integration by parts, it turns out to be

$$= \int_{R^n} \int_a^b \theta \big(L(X_{RSS} \mid \theta) g(\theta) \big) \big|_a^b \, dX_{RSS} - \int_{R^n} \int_a^b L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS} \qquad \text{(A-3)}$$

$$= -\int_{R^n} \int_a^b L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS} [\because g(a) = g(b) = 0] \qquad \text{(A-4)}$$

Subtracting (A-2) from (A-1) we have

$$\int_{R^n} \int_a^b \{\hat{\theta}(X_{Rss}) - \theta\} \left(\frac{\delta}{\delta\theta} L(X_{Rss} \mid \theta) g(\theta) \right) d\theta dX_{RSS} \qquad \text{(A-5)}$$

$$= \int_{R^n} \int_a^b L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS}$$

By using the Fubini's theorem we have,

$$\int_a^b \int_{R^n} \big(L(X_{Rss} \mid \theta) dX_{RSS} \big) g(\theta) d\theta = 1 \qquad \text{(A-6)}$$

Multiplying and dividing by $L(X_{Rss} \mid \theta) g(\theta)$, left hand of the equation-(A-5) gives

$$\int_{R^n} \int_a^b \{\hat{\theta}(X_{Rss}) - \theta\} \left(\frac{\delta}{\delta\theta} \log\big(L(X_{Rss} \mid \theta) g(\theta) \big) \right) L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS} = 1$$

$$\left(\int_{R^n} \int_a^b \{\hat{\theta}(X_{Rss}) - \theta\} \left(\frac{\delta}{\delta\theta} \log\big(L(X_{Rss} \mid \theta) g(\theta) \big) \right) L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS} \right)^2 = 1$$

By using the Cauchy-Schwartz inequality we have

$$\left(\int_{R^n} \int_a^b \{\hat{\theta}(X_{Rss}) - \theta\}^2 L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS} \right) \left(\int_{R^n} \int_a^b \left(\frac{\delta}{\delta\theta} \log\big(L(X_{Rss} \mid \theta) g(\theta) \big) \right)^2 L(X_{Rss} \mid \theta) g(\theta) d\theta dX_{RSS} \right) \geq 1$$

(A-7)

Here

$$\int_{R^n}\int_a^b \{\hat{\theta}(X_{Rss})-\theta\}^2 L(X_{Rss}\mid\theta)g(\theta)d\theta dX_{RSS}$$
$$=\int_a^b\left(\int_{R^n}\{\hat{\theta}(X_{Rss})-\theta\}^2 L(X_{Rss}\mid\theta)dX_{RSS}\right)g(\theta)d\theta$$
$$=E_\theta[E_{X_{Rss}|\theta}\{\hat{\theta}(X_{Rss})-\theta\}^2] \quad \text{(A-8)}$$

and

$$\int_{R^n}\int_a^b\left(\frac{\delta}{\delta\theta}\log\left(L(X_{Rss}\mid\theta)g(\theta)\right)\right)^2 L(X_{Rss}\mid\theta)g(\theta)d\theta dX_{RSS}$$
$$=\int_{R^n}\int_a^b\left(\frac{\delta\log\left(L(X_{Rss}\mid\theta)\right)}{\delta\theta}+\frac{\delta\log\left(g(\theta)\right)}{\delta\theta}\right)^2 L(X_{Rss}\mid\theta)g(\theta)d\theta dX_{RSS}$$
$$=\int_{R^n}\int_a^b\left(\frac{\delta\log\left(L(X_{Rss}\mid\theta)\right)}{\delta\theta}\right)^2 L(X_{Rss}\mid\theta)g(\theta)d\theta dX_{RSS}$$
$$+\int_{R^n}\int_a^b\left(\frac{\delta\log g(\theta)}{\delta\theta}\right)^2 L(X_{Rss}\mid\theta)g(\theta)d\theta dX_{RSS}$$
$$+2\int_{R^n}\int_a^b\left(\frac{\delta\log\left(L(X_{Rss}\mid\theta)\right)}{\delta\theta}\right)\left(\frac{\delta\log\left(g(\theta)\right)}{\delta\theta}\right)L(X_{Rss}\mid\theta)g(\theta)d\theta dX_{RSS}$$
$$=\int_a^b\left(\int_{R^n}\left(\frac{\delta\log\left(L(X_{Rss}\mid\theta)\right)}{\delta\theta}\right)^2 L(X_{Rss}\mid\theta)dX_{RSS}\right)g(\theta)d\theta$$
$$+\int_a^b\left(\frac{\delta\log\left(g(\theta)\right)}{\delta\theta}\right)^2\left(\int_{R^n}L(X_{Rss}\mid\theta)dX_{RSS}\right)g(\theta)d\theta$$
$$+2\int_a^b\left(\int_{R^n}\frac{\delta\log\left(L(X_{Rss}\mid\theta)\right)}{\delta\theta}L(X_{Rss}\mid\theta)dX_{RSS}\right)\frac{\delta\log\left(g(\theta)\right)}{\delta\theta}g(\theta)d\theta$$
$$=E_\theta\left(I_{RSS}(\theta)\right)+I_g(\theta) \quad \text{(A-9)}$$

Assumption-1 implies that $\int_{R^n}\frac{\delta}{\delta\theta}\log\left(L(X_{Rss}\mid\theta)\right)L(X_{Rss}\mid\theta)dX_{RSS}=0$ and $\int_{R^n}L(X_{RSS}\mid\theta)dX_{RSS}=1$

Putting (A-8) and (A-9) in (A-7) we get,

$$E_\theta[E_{X_{RSS}|\theta}\{\hat{\theta}(X_{RSS})-\theta\}^2]\geq\left[E_\theta\left(I_{RSS}(\theta)\right)+I_g(\theta)\right]^{-1}$$

Table 1: Relative Efficiency (RE), reduction in sample size (in %) and required sample size for parameter λ under RSS as compare to SRS for different values of a and b, and set size $s = 3$

Relative Efficiency						
		a				
		1	3	5	7	9
b	1	1.28	1.11	1.06	1.03	1.02
	3	1.67	1.42	1.26	1.17	1.12
	5	1.79	1.64	1.47	1.34	1.25
	7	1.81	1.76	1.64	1.5	1.39
	9	1.81	1.8	1.74	1.63	1.52
Reduction in Sample Size (in %)						
b	1	21.88	9.91	5.66	2.91	1.96
	3	40.12	29.58	20.63	14.53	10.71
	5	44.13	39.02	31.97	25.37	20.00
	7	44.75	43.18	39.02	33.33	28.06
	9	44.75	44.44	42.53	38.65	34.21
Required Sample Size						
b	1	2	3	3	3	3
	3	2	2	2	3	3
	5	2	2	2	2	2
	7	2	2	2	2	2
	9	2	2	2	2	2

Table 2: Relative Efficiency (RE), reduction in sample size (in %) and required sample size for parameter λ under RSS as compare to SRS for different values of a and b, and set size $s = 5$

Relative Efficiency						
		a				
		1	3	5	7	9
b	1	1.55	1.23	1.12	1.07	1.04
	3	2.34	1.83	1.52	1.34	1.23
	5	2.58	2.29	1.94	1.68	1.5
	7	2.61	2.52	2.27	2.00	1.78
	9	2.62	2.6	2.47	2.26	2.04
Reduction in Sample Size (in %)						
b	1	35.48	18.7	10.71	6.54	3.85
	3	57.26	45.36	34.21	25.37	18.70
	5	61.24	56.33	48.45	40.48	33.33
	7	61.69	60.32	55.95	50.00	43.82
	9	61.83	61.54	59.51	55.75	50.98
Required Sample Size						
b	1	3	4	4	5	5
	3	2	3	3	4	4
	5	2	2	3	3	3
	7	2	2	2	2	3
	9	2	2	2	2	2

Table 3: Relative Efficiency (RE), reduction in sample size (in %) and required sample size for parameter λ under RSS as compare to SRS for different values of a and b, and set size $s = 10$

Relative Efficiency						
		a				
		1	3	5	7	9
b	1	2.24	1.51	1.26	1.15	1.1
	3	4.01	2.88	2.17	1.76	1.52
	5	4.56	3.9	3.13	2.53	2.12
	7	4.63	4.41	3.86	3.26	2.76
	9	4.64	4.59	4.31	3.84	3.35
Reduction in Sample Size (in %)						
b	1	55.36	33.77	20.63	13.04	9.09
	3	75.06	65.28	53.92	43.18	34.21
	5	78.07	74.36	68.05	60.47	52.83
	7	78.4	77.32	74.09	69.33	63.77
	9	78.45	78.21	76.8	73.96	70.15
Required Sample Size						
b	1	4	7	8	9	9
	3	2	3	5	6	7
	5	2	3	3	4	5
	7	2	2	3	3	4
	9	2	2	2	3	3

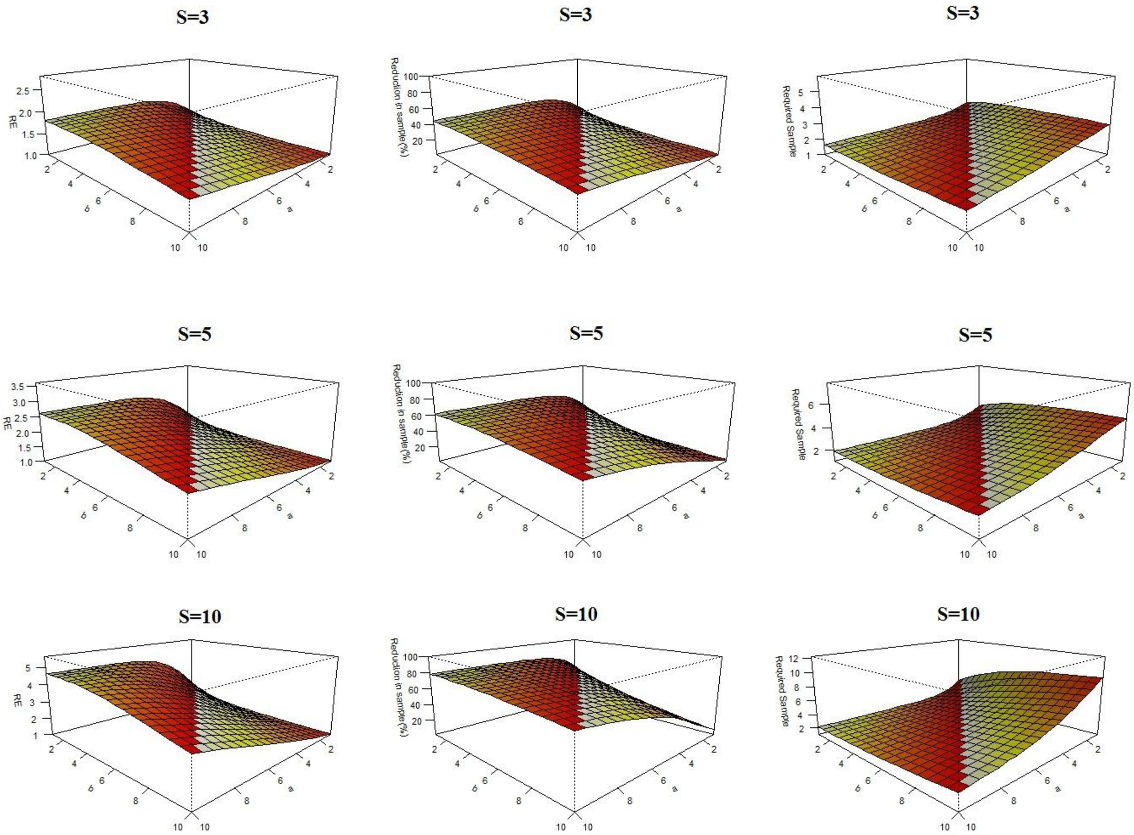

Figure 1: Relative Efficiency (RE), reduction in sample size (in %) and required sample size for parameter λ under RSS as compare to SRS for different values of a and b, and set size $s = \{3, 5, 10\}$.