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"Development and Application of FBBFBEM Solution for 3-DComplex- Shaped Domain"

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Abstract

In this article, for solving the 3-D thermal conduction problem an effort has been by using the boundary integral equation formulation and BEM. Multidimensional non uniform bodies which are in complex shape are addressed by triangular patch modeling method. The arbitrary geometry of the body is divided into elements which are of triangular patches by triangulation algorithm. A numerical solution is developed by defining the basis functions on the face of the triangles generated in the triangular patch modeling, in contrast to defining it on the nodes as followed in the regular mesh generation technique like BEM, FVM and FEM solutions. The temperature distribution from the surface of the hot body is plotted for understanding the nature of its behaviors. Also the convergence study is conducted to get the solution convergence towards the exact solution for the case of a sphere. Other geometries like cube, cylinder and cone are also treated and analyzed for the field variable.

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Keywords: Boundary Integral Equations; Thermal Conduction; Face Based Basis Functions (FBBF); Error Analysis.

1. Introduction

The problems in science and engineering are usually reduced to a mathematical equations i.e. set of coupled partial differential equations (PDEs). It is not easy to obtain their analytical solution, particularly in non-linear and complex-shaped cases, and discrete approximate mathematical approach have to be employed accordingly.Heat conduction is one of three modes of heat transfer and play a crucial role in the engineering design as well as in science application like heat sinks, heat engines, electronic cooling etc. The heat conduction is part of almost all the

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heat transfer processes and it is difficult to find a heat transfer application without the presence of heat conduction. Heat conduction is governed by the Fourier law of heat conduction.

i.e, $$Q = KA\frac{dT}{dx} \tag{1}$$

Where Q is the heat flow rate by conduction ($W \cdot m^{-2}$), k is the thermal conductivity of body material ($W \cdot m^{-1} \cdot K^{-1}$), A is the cross-sectional area normal to direction of heat flow (m^2) and dT/dx is the temperature gradient ($K \cdot m^{-1}$). As per the Fourier's governing law, heat transfer into the body or out of the body through heat conduction process depends on the geometry of the body, thermal conductivity and temperature gradient. The heat transfer phenomenon through conduction is usually represented in the differential equation form along with the boundary conditions. The boundary conditions can be either Dirichlet (essential) boundary conditions or Neumann (natural) boundary conditions or mixed boundary conditions. In Dirichlet boundary conditions, temperature T is specified on the boundary; whereas in Neumann boundary conditions, heat flux Q is specified. In case of mixed boundary conditions, temperature is specified on a part of the boundary and heat flux is specified on remaining part of the boundary. The BEM requires discretization of the surface rather than discretizing the entire continuum as in FEM. The BEM method is used for more complex mathematical calculations. The computer code written for BEM is easier and simpler to use as compared to FEM. The governing equations are solved on these known geometrical shapes and the temperatures are determined. When the governing equations are expressed in the form of differential equations, then solution domain is the volume of the geometry in which the temperature distribution is to be found. The numerical solution of heat conduction problems are classified into two categories, such as:

- Volume domain approach Method
- Boundary approach Method

The volume domain approach consists of the popular FDM and FEM. These methods discretize the whole domain into elements. Those elements on the boundaries are thus analyzed together with the interior points to solve for the temperature. The other numerical methods that use the method of solving the differential equation of the heat conduction are Finite Volume Methods (FVM) and Spectral Methods(SM) etc. The main drawback in using the differential equation based numerical solution methods is, if the solution domain is an open domain or very intricate domain, then the number of basic elements that are required to be generated is enormously high for an accurate solution, which makes the computational time very intensive. Example of these kind of applications include heat transfer in geometries having very intricate shapes by closed domain problems, open domain problems like thermal distribution in the sea when ship is moving around so that one can estimate the thermal effects on the marine life etc.

In order to alleviate some of these difficulties like computational cost when dealing with the open domain problems, one can use the integral equation formulation instead of differential equation formulation. In integral equation formulation, only the boundary of the geometry is discretized if it is a open domain problem or a closed domain problem and hence number of elements generated in the boundary is limited. The numerical solution method for the boundary integral equation methods is known as boundary element method. Modern computational techniques facilitate solving problems with imposed boundary conditions using different numerical methods. Boundary element method is quite popular in dealing with the structural analysis and is a well-known numerical method in that area [2, 3]. In the BEM solution proposed in [1,2], the basis function are defined with respect to the nodes similar to the FEM. Authors in ref [3,4] introduced a novel method of defining the basis function on the triangular face of the elements in contrast to defining it with respect to the nodes. In the solution methodology proposed by authors [3,4], there is no concept of nodes required to define the basis functions. Later the basis functions were defined on the edges [5] and then on the nodes [6]. But the basis functions defined in the research work [1] is quite different from what is used in ref [2, 3]. However the methods proposed in ref [3-6] are applied to the acoustic scattering problems rather than heat conduction problem. In this research work it is proposed to define the face based basis functions on patches to solve the heat conduction problem which involves sophisticated mathematical technique. The basis functions defined in this work are similar to the one proposed in ref [1,3] but for a heat conduction problem rather than for an acoustic problem. Other numerical methods those are available in [12-22] to address the problem of heat conduction in thermal science and engineering.

2. Boundary Element Method

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For an equation resulting from equations, which is in the form of

$$\boldsymbol{L}\,\boldsymbol{f} = \boldsymbol{g} \tag{2}$$

Where $\boldsymbol{L}$: Linear operator, $\boldsymbol{g}$: Known function resulting from the forcing function

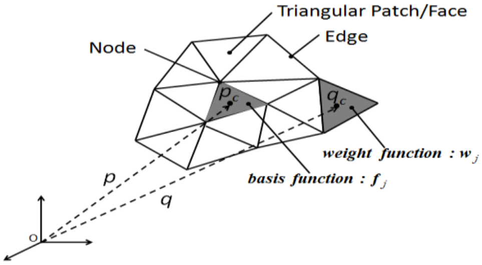
$\boldsymbol{f}$: Unknown function to be determined,

The solution can be derived as follows:

Let the unknown function $\boldsymbol{f}$ be approximated by a set of known functions $\boldsymbol{f}_j$, $j = 1,2,\ldots, N$

$$\boldsymbol{f} = \sum_{n=1}^{N} \beta_j \boldsymbol{f}_j \quad \text{where } \boldsymbol{f}_j \text{: Basis functions in the domain of } \boldsymbol{L} \tag{3}$$

β_j : are scalar coefficients to be determined.



Substituting Eq. 3 into Eq. 2

$$\sum_{n=1}^{N} \beta_j \boldsymbol{L}\,\boldsymbol{f}_j = \boldsymbol{g} \tag{4}$$

where the equality is usually approximate.

Let $\boldsymbol{w}_i$ testing functions in the range of $\boldsymbol{L}$. Now, taking the inner product of Eq. 4 with each w_i and using the linearity of inner product defined as $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_S \boldsymbol{f} \bullet \boldsymbol{g}\, ds$, we obtain a set of linear equations, given by

$$\sum_{n=1}^{N} \beta_j \langle \boldsymbol{w}_i, \boldsymbol{L}\,\boldsymbol{f}_j \rangle = \langle w_i, \boldsymbol{g} \rangle \quad i = 1,2,\ldots, N \tag{5}$$

The set of equations in Eq. 4 may be written in the matrix form as

$$\boldsymbol{ZX} = \boldsymbol{Y} \tag{6}$$

This can be solved for $\boldsymbol{Z}$ using any standard linear equation solution methodologies. The simplicity, accuracy and efficiency of the BEM lies in choosing proper set of basis/testing functions and applying to the problem at hand. In this work, we propose a special set of basis functions and a novel testing scheme to obtain accurate results.

3. Mathematical Formulation

Let T is the scalar thermal potential satisfying the Helmholtz differential equation $\nabla^2 T + k^2 T = 0$ for the time harmonic waves present in the region exterior to the surface S of the body. Another condition that the thermal potential must satisfy is the appropriate boundary conditions on the surface S of the body.

Using the potential theory and the free space Green's function, the scattered thermal potential T^s may be defined as

$$T^s = \int_S \sigma(\boldsymbol{p})\, G(\boldsymbol{p}, \boldsymbol{q})\, ds' \tag{7}$$

In the above three equation, σ is the source density function dependent of p over the surface of the body, $p$is the position vector of source points, with respect to a global co-ordinate system O, q is the position vector of observation points, with respect to a global co-ordinate system O.

$G(\boldsymbol{p},\boldsymbol{q})$ is the free space Green's function. $G(\boldsymbol{p},\boldsymbol{q}) = \dfrac{e^{k|p-q|}-1}{|p-q|}$

For a body, that has the fixed temperature defined on the surface of the body, the total thermal potential is zero, i.e.

$$\Phi^i + \Phi^s = 0 \tag{8}$$

Hence,

$$\int_S \sigma(\boldsymbol{p})G(\boldsymbol{p},\boldsymbol{q})ds' = -T^i \tag{9}$$

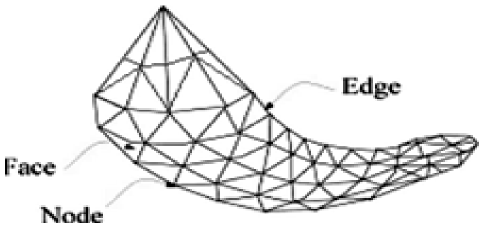


Fig.1 Triangular patch modeling of an arbitrarily shaped 3-D body.

4. Numerical Solution Procedure

The numerical solution procedure to solve the Eq.(3).by using the basis functions is explained below.
Testing Eq. 13 with a testing function w_m , results in

$$\left\langle w_q, \int_S \sigma(\boldsymbol{p})G(\boldsymbol{p},\boldsymbol{q})ds' \right\rangle = \left\langle w_q, T^i \right\rangle \tag{10}$$

Using the inner product definition, 4, Eq. 5 can be written as

$$\int_S w_q \int_S \sigma(\boldsymbol{p})G(\boldsymbol{p},\boldsymbol{q})ds'\, ds = \int_S w_q T^i ds \tag{11}$$

The weighing function can be defined as

$$w_j = \begin{cases} 1 & q_j \in S \\ 0 & otherwise \end{cases} \tag{12}$$

Let

$$\sigma(\boldsymbol{p}) = \sum_{n=1}^{N} \beta_j \boldsymbol{f}_j \tag{13}$$

Approximating the Eq. 6 over the source triangular patch,

$$\int_S w_q \int_S \sum_{j=1}^{N} \beta_j f_j G(p_c, q) ds' ds = \int_S w_q T^i ds \tag{14}$$

Approximating the integration over the field triangular patch at the centroids, Eq. 5 become

$$\int_S w_i \int_S \sum_{j=1}^{N} \beta_j f_j G(p_c, q_c) ds' ds = \int_S w_i T^i ds \tag{15}$$

Let the basis function be defined as

$$f_j = \begin{cases} 1 & p_j \in S \\ 0 & otherwise \end{cases} \tag{16}$$

$$A_i \sum_{j=1}^{N} \beta_j f_j G(p_c, q_c) ds' = A_i T^i \tag{17}$$

where A_i is the area of field triangular patch. This approximation is justified because the domains are sufficiently small, which is a necessary condition to obtain accurate solution using BEM. For a pulse function defined on the source triangular patch, it results in a system of linear equations, which can be represented in the matrix form as

$$\boldsymbol{Z} \ \boldsymbol{X} = \boldsymbol{Y} \tag{18}$$

where $\boldsymbol{Z}$ is the impedance matrix of the single layer formulation of size $N_f \ X \ N_f$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the column vectors of size N_f .The elements of $\boldsymbol{Z}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are given below.

$$\boldsymbol{Z}_{m,n} = \int_S G(p_c, q_c) ds' \tag{19}$$

and $\boldsymbol{Y_m} = T^i$ (20)

Where q_c is the position vector to the centroid of the field triangular patch, p_c is the position vector to the centroid of source triangular patch. Once the matrix Z is determined and vector Y is calculated, vector X can be calculated using any standard linear equation solvers. The salient features of the proposed method are less computational time and less storage memory

5. Results and Discussion

The FBBF BEM solution procedure mentioned in Sec. 2 followed to obtain the temperature diffusion solution exterior to the sphere of radius 1m. The outside surface of the sphere is maintained at a temperature of 100^{o}C. The sphere is discretized with triangular patch modelling with 4 different mesh sizes. Outer surface of the sphere is divided into 10 parts each in the polar and azimuthal directions resulting in a total of 180 triangular patches as shown in Fig.2.(a) Similarly by dividing the sphere 12, 15 and 20 parts in the polar and azimuthal directions, it results in a total of 264, 420 and 760triangular patches respectively. Higher the number of patches, the more accurate the solution is.

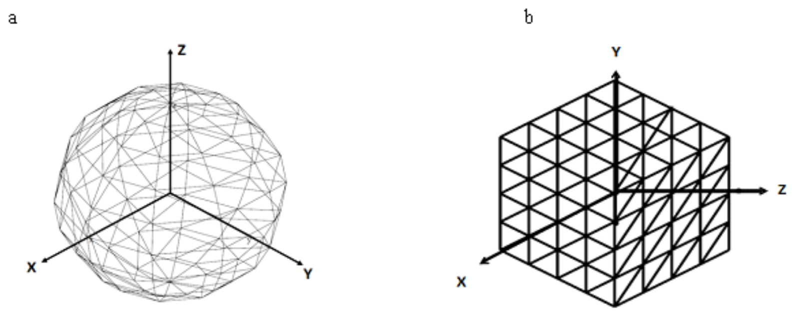


Fig.2.Triangular Patch Model of a (a) Sphere; (b) Cube

Fig.2 (b) shows the triangular patch modelling of the cube. Cube is divided into the triangular patches by dividing each edge into 4 parts there by creating a total of 192 patches. The surface of the cube is maintained at a temperature of 100^0C and temperature distribution is predicted using the procedure mentioned above.

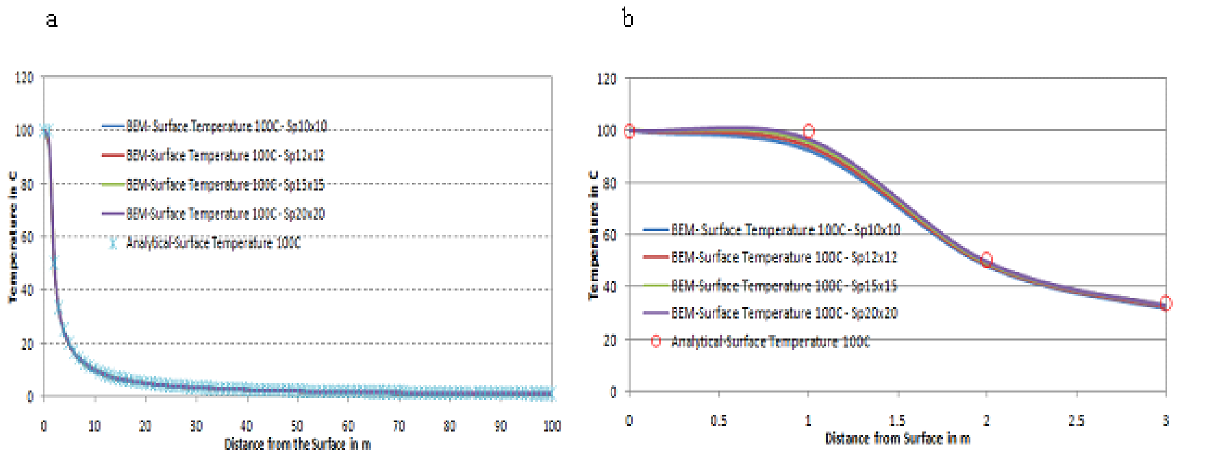


Fig.3.(a)and(b)Temperature Distribution outside a Sphere of Radius 1m, Maintained at a Temperature of 100^0C up to a Distance of 100m.and 3m.

Fig. 3(a) shows the temperature distribution outside a sphere of radius 1m, maintained at a temperature of 100^0C upto a distance of 100m. The heat conduction problem considered here is a steady state model and hence it is independent of the thermal conductivity of the surrounding medium. The temperature falls very rapidly in the close vicinity of the hot surface and there after it changes very slowly and diffuses gradually as the distance increases.

Fig.3 (b) shows the temperature distribution outside a sphere of radius 1m, maintained at a temperature of 100^0C up to a distance of 3m. It shows that the temperature falls dramatically by 50% at a distance of the 2m from the surface of the body. That is at distance of twice the radius of the sphere the temperature fall by 50%. The numerical results improve and converge towards the exact solution as the number of triangular patches increase from 180, 264, and 420 and to 760. The exact solution for the temperature outside the sphere is given by Tb/r. Where Tb is the temperature of the body and r is the distance from the surface of the sphere.

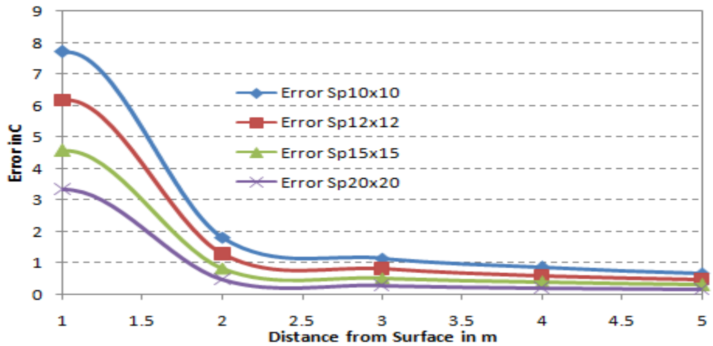


Fig.4.Temperature Difference w.r.t Exact Solution outside a Sphere of Radius 1m, Maintained at a Temperature of 100^0C up to a Distance of 5m.

Fig. 4 shows the temperature difference outside a sphere of radius 1m, maintained at a temperature of 100^0C up to a distance of 5m. For the sphere with 120 patches (sp10x10) the error in the temperature is around 7.5^0C at a distance of 1m. But it comes down to 3.2^0C as the number of patches increased to 760 patches. Similarly, gap between the predicted numerical results and the exact solution reduces at the other points also as the number of patches on the model increases. The solution can further be converged towards the exact solution by increasing the number of patches beyond 760 patches. However it needs more computational resources for the solution.

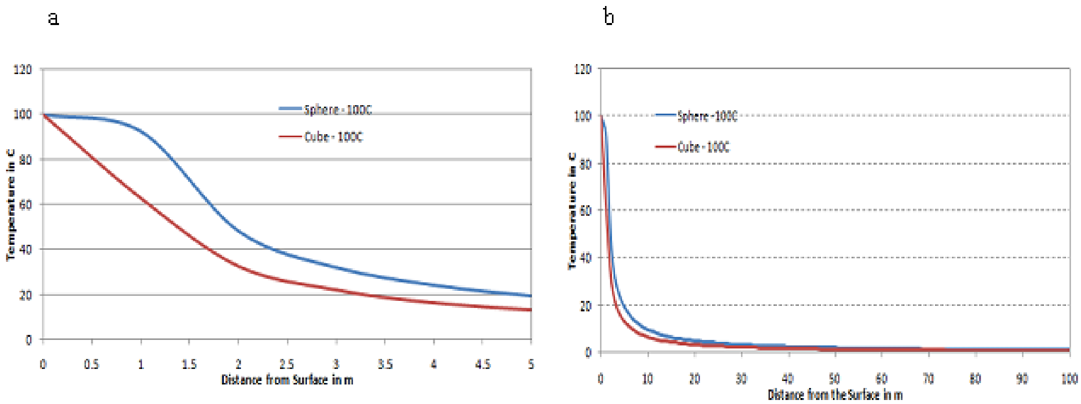


Fig.5.(a)and (b)Temperature Distribution outside a Cube Having Each Side of 1m, Maintained at a Temperature of 100^0C upto a Distance of 5mand 100m

Fig. 5(a) and (b) shows the temperature distribution along the X-Axis up to a distance of the 5m and 100m. The temperature distribution of the cube is compared with that of the sphere. Sphere shows higher temperature since there is contribution of the bigger exposed surface area from the sphere than that in the cube which follows the basic law of thermal conduction.

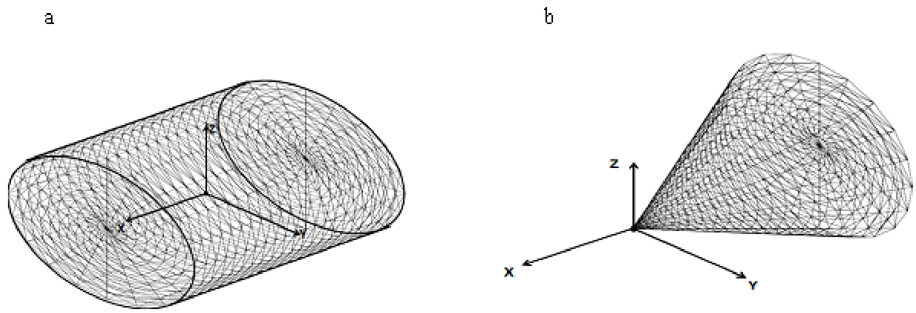


Fig.6.Triangular patch model of a (a)Cylinder; (b)Cone

Fig. 6(a) shows the triangular patch modelling of the cylinder. Cylinder considered of radius 1m and height of 2m. The circumference of the cylinder is divided into 20 equal parts and height also into 20 equal parts. The cylinder is closed with two end caps. The radiuses of the caps are divided into 10 equal parts. This results into a total of 1560 triangular patches. The surface of the cylinder is maintained at a temperature of 100C. Temperature is predicted using the procedure mentioned above.

Fig.6.(b) shows the triangular patch modelling of the cone. Cone is of radius 1m and height of 2m. The circumference of the Cone is divided into 20 equal parts and slant edges also into 20 equal parts. The Cone is closed with a cap at the bottom. The radius of the cap is divided into 10 equal parts. This results into a total of 1160 triangular patches. The surface of the cone is maintained at a temperature of 100^0Cand temperature behavior is predicted using the procedure mentioned above.

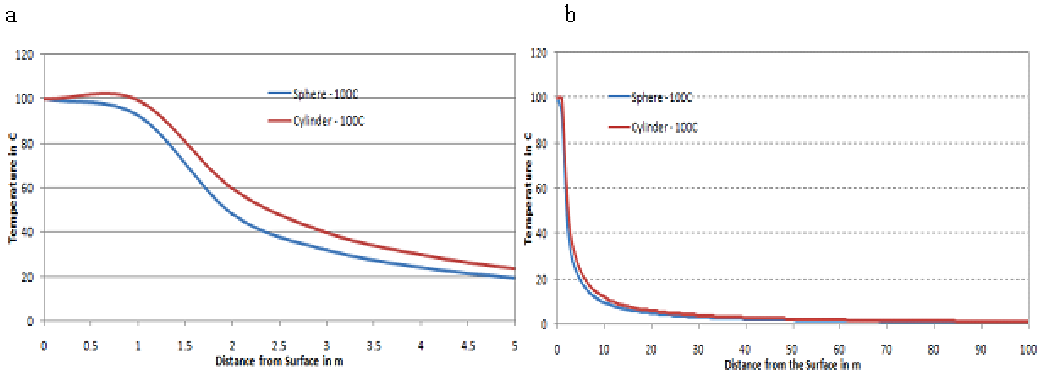


Fig.7.(a)And (b) TemperatureDistribution outside a Cylinder Having Radius of 1m and Height 2m, Maintained at a Temperature of 100^0C up to a Distance of 5m and 100m.

Figs.7. (a) And (b) show the temperature distribution along the X-axis for distances up to 5m and 100m respectively. Temperature distribution resulting from the cylinder is greater than that of the sphere. This is due to the reason that, the size of the cylinder, with higher surface area, is bigger than that of the sphere. Hence there is more heat contributed from the cylinder than the sphere.

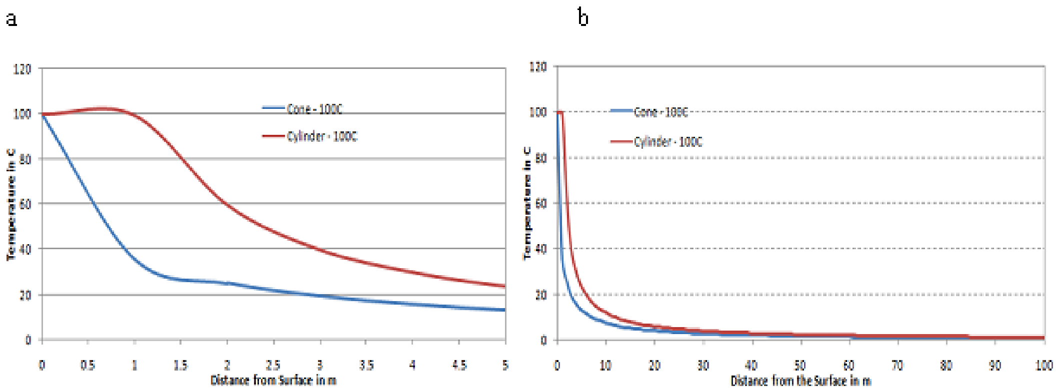


Fig.8. (a) and (b)Temperature Distribution outside a Cone Having Radius of 1m and Height 2m, Maintained at a Temperature of 100^0C up to a Distance of 5m and 100m

Fig.8.(a) and (b) show the temperature distribution along the X-axis for distances up to 5m and 100m respectively. Temperature distribution resulting from the cone is less than that of the cylinder. This is due to the reason that, the size of the cylinder, with higher surface area, is bigger than that of the cone. Hence there is more heat contributed from the surface of the cylinder than the cone. These are some examples run to demonstrate the capability of the patch based basis function. The examples are chosen to test the capability of the patch based basis function having

rounded surfaces, sharp edges, sharp corners etc.

6. Conclusions

In this research work, we demonstrate efficacy of the proposed BEM method to solve the complex shaped steady state heat conduction problem. The face based basis functions (FBFB) are defined directly on the triangular face of the patch rather than on the nodes. The effort has been made to obtain the solution for 4 models of the sphere and the results are compared with the exact solution. The solution converges towards the exact solution and the error narrows down from 7.5^0C to 3.2^0C as the number patches increased from 120 to 760. The exact solution is available only for the spherical geometry and hence it is possible to compare the numerical results with that of the exact solution. Furthermore, it is demonstrated that for geometries which are irregular in shape, the exact solutions are not available and hence numerical solution procedure which is validated for the spherical case can be applied. Geometries of cube having sharp edges and sharp corner, cylinder with circular edge, cone with circular edge and a sharp corner are solved for temperature distribution. Out of all irregular geometries, the temperature diffusion from the cylinder is quietly higher than any geometry because of the larger size and surface area. In conclusion, the proposed FBBF BEM is computationally efficient, robust, stable and convergent with respect to increasing number of patches.

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