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Number Of Level Crossings Of Cauchy Algebraic Polynomials

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Abstract

A Cauchy algebraic polynomial is a random algebraic polynomial

$$f_n(x,w) = \sum_{k=0}^{n} a_k(w)x^k, x \in (-\infty,\infty),$$

whose coefficients are independent real-valued random variables with a common Cauchy distribution then for every large enough integer n_0, $P_r\left\{\underset{n>n0}{Sup} N_n(w) > \mu'(\log n)^2\right\} < \mu''/n^{s-2-\beta}$

where s is a finite number greater than 2+β; 0< β<1 and μ's are positive constants.For this theorem we get, for s>3, a probability less than $\mu''/n^{1-\beta}$

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1. Introduction

A Cauchy algebraic polynomial is a random algebraic polynomial,

$$f_n(x,w)=\sum_{k=0}^{n} a_k(w)x^k,\ x\in(-\infty,\infty), \qquad (1.1)$$

whose coefficients are independent real-valued random variables with a common Cauchy distribution. Let $N_n(w)$ and $E(N_n)$ denote the number of real roots and its mathematical expectation, of the random equations $f_n(x,w)=0$, Logan and Shepp [1] have shown that

$$E(N_n)\sim c\log n \qquad (n\to\infty),$$

where

$$c=\frac{8}{\pi^2}\int_0^{\infty}\frac{ze^{-z}}{z-1+2e^{-z}}dz,$$

Using this one gets that

$$P_r\{N_n>\mu'(\log n)^2\}<\mu''/(\log n)$$

where μ's are positive constants.

Samal and Mishra [2] have considered the upper bound of $N_n(w)$ where the coefficients are identically distributed independent random variables with a common characteristic function exp (-Ctα), where C is a positive constant and $1\leq\alpha\leq 2$. They have obtained that

$$P_r\left\{\underset{n>n0}{Sup}\frac{N_n(w)}{(\log n)^2}>\mu'\right\}<\mu''/n^{3\alpha-2-\beta}$$

where $0<\beta<1$. This covers the Cauchy case for α=1

In [2], Samal and Mishra have employed the Inversion Formula to obtain certain probability estimates. In this note we show that in case of Cauchy polynomials with identically distributed coefficients such estimates can be obtained by using the density function. We show that the probability that

$$\left\{ \underset{n>n0}{Sup} N_n(w) \right\} > \mu'(\log n)^2$$

is smaller than $\mu''/n_0^{1-\beta}$ in the Cauchy case. In fact, the probability we have estimated can be made *arbitrarily* small. Precisely, we prove the following theorem:

2. THEOREM

Let *fn(x,w)* be a Cauchy algebraic polynomial of degree n. Then for every large enough integer n_0,

$$P_r \left\{ \underset{n>n0}{Sup} N_n(w) > \mu'(\log n)^2 \right\} < \mu''/n^{s-2-\beta}$$

where s is a finite number greater than 2+β; 0< β<1 and μ's are positive constants.

For this theorem we get, for s>3, a probability less than $\mu''/n^{1-\beta}$

In the proof of the theorem we need the following lemma which is a well-known result.

2.1 Lemma

Let $X(w) = \sum_{k=0}^{n} a_k(w) x^k$

Be a continuous random variable with probability density $p_n(w)$. Then

$$U = X^2(w) = \left\{ \sum_{k=0}^{n} a_k(w) x^k \right\}^2$$

is a continuous random variable with probability density

$$PU(w) = \left\{ \frac{1}{2\sqrt{w}} \left\{ P_n(\sqrt{w}) + P_n(-\sqrt{w}) \right\} \right\} \text{ for w>0.}$$

In what follows we have assumed n to be sufficiently large for the inequalities to hold and we have used μ's to denote positive constants not necessarily having the same value from one place of occurrence to the other.

Proof of the theorem

We refer to the proof of Samal and Mishra [2] and indicate only the important steps or modifications, always with α=1 for the Cauchy case. It follows as in [2; p.600] that

$$P_r\left\{|a_k(w)| < (n+1)^2, k = 0,1,2,............n\right\}$$
$$> 1 - \mu(n+1)^{s-1} \qquad (2.1)$$

for α=1,

So

$$\left\{\underset{|z|\le 1+2/n}{\mathrm{Max}}\left|f(z) \le e^z (n+1)^{z+1}\right|\right\} \qquad (2.2)$$

with a probability greater than $1 - \mu/(n+1)^{s-1}$

Again, the probability density of $a_k(w)$, k=0,1,………..n is given by

$$P_w = \frac{1}{\pi}\frac{c}{c^2 + w^2},$$

c being a positive constant, We put it as

$$P_w = \begin{cases} 0(1/2^{2n}), & -2^n \le w \le 2^n \text{ for large n} \\ 0 \end{cases} \qquad (2.3)$$

otherwise

So that the probability density of $a_k(w)\ X_m^k$

$$\left(1/x^k{}_m\right)P(w/x^k{}_m) = \begin{cases} 0(1/2^{2n}), & -2^{nx^k{}_m} \le w \le 2^{nx^k{}_m} \\ 0 \end{cases}$$

otherwise

Since $\frac{1}{2} \leq X_m \leq 1$, we have $\left(-2^n X^k{}_m, 2^n X^k{}_m\right) \subset \left(-2^n, 2^n\right)$. Thus, the probability density of

$a_k(w) X^k{}_m$ in $\left(-2^n, 2^n\right)$ is $0\left\{1/2^{2n} X^k{}_m\right\}$.

Let $P_n(w)$ be the probability density of

$$\left\{\sum_{k=0}^{n} a_k(w) x_m{}^k\right\}$$

We have that

$$P_1(w) = \int_{-2\pi}^{2\pi} p(z)(w-z)dz$$

$$= \frac{\mu}{2^{2n}} \int_{-2^n}^{2n} (w-z)dz$$

$$= \frac{\mu}{(2^{2n})^2} \cdot \frac{2^{n+1}}{X_m}$$

$$P_2(w) = \int_{-2\pi}^{2\pi} p_1(z)(w-z)dz$$

$$= \frac{\mu}{(2^{2n})^3} \cdot \frac{2^{2(n+1)}}{X^{1+2}{}_m}$$

and in general

$$P_n(w) = \frac{\mu}{(2^{2n})^{n+1}} \cdot \frac{2^{2(n+1)}}{X^{1+2+.....+n}{}_m}$$

$$= \frac{\mu}{(2^{2n} X_m)^{n(n+1)/2}} \qquad (2.4)$$

Since $1 \leq X_m \leq 1\!\!/_2$.

Using this, the lemma gives

$$P_u(w) = \begin{cases} 0(1/\sqrt{w}), \\ 0 \end{cases} \qquad \text{for } w>0$$

$$\text{for } w \le 0 \qquad (2.5)$$

Now

$$\begin{aligned} & p\left(\left|f_n(x_m)\right| < e\right) \\ & \le p\left(\left\{\left|f_n(x_m)\right|\right\}^2 < e^2\right) \\ & = P(U < \in^2) = \int_{-\infty}^{\in^2} p\nu(w)dw \\ & = \int_{-\infty}^{\in^2} p\nu(w)dw \\ & \le \mu \int_{-\infty}^{\in^2} \frac{1}{\sqrt{w}} dw \\ & = 2\mu s \end{aligned} \qquad (2.6)$$

Putting $\in = 1/n^{s-1}$, this gives

$$P\left\{\left|f_n(x_m)\right| < \frac{1}{n^{s-1}}\right\} \le 2\mu/n^{s-1} \qquad (2.7)$$

Using (2.1) and (2.7) we obtain that with a probability greater than $1-\left(\mu/n^{s-1}\right)$ the number of zeros of f_n (x) in C_m is at most μ(logn). Considering all the circles the total number of zeros inside all the circles C0, C1,………Ck, Cplogn is at most μ(logn)2m with a probability of measure at least

$$\left\{1-\mu'(\log n)/n^{s-1}\right\} > 1-\mu''/n^{s-1-\beta}; 0<\beta<1.$$

The number of zeros in C^* is at most $\mu(\log n)$ with probability larger than $1-\mu/n^{s-1-\beta}$.

Thus for $n > n_e, N_n(w) > \mu(\log n)^2$, with probability less than

$$\sum_{n=n_0+1}^{\infty}\left(\mu / n^{s-1-\beta}\right) < \mu'' / n^{s-2-\beta}; 0 < \beta < 1,$$

where s is a finite number greater than 2+β

This completes the proof.

3. CONCLUSION

Considering a Cauchy random algebraic polynomial of the form

$$f_n(x,w) = \sum_{k=0}^{n} a_k(w)x^k, x \in (-\infty,\infty),$$

whose coefficients are independent real-valued random variables with a common Cauchy distribution then for every large enough integer n_0,the the expected number of zeros of the above polynomial

$$P_r\left\{\underset{n>n0}{Sup} N_n(w) > \mu'(\log n)^2\right\} < \mu'' / n^{s-2-\beta}$$

where s is a finite number greater than 2+β; 0< β<1 and μ's are positive constants.For this theorem we get, for s>3, a probability less than $\mu'' / n^{1-\beta}$

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